Centre of mass

(Determination of centre of mass)

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Definition of centre of mass

Centre of mass of a system is the point where the entire mass of the body is assumed to be concentrated.

Examples

- Balancing a book at the tip of a finger
- Spinning a basket ball at the tip of a finger
- Carrying luggage comfortably when distributed evenly in both hands
- High jump of athletes
- Ball tracking mechanism

Determination of coordinates of COM of discrete particles

Centre of mass of a system consisting of discrete point masses is given by the relation

$$\boxed{x_{\text{cm}} = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3 \ldots}{m_1 + m_2 + m_3 \ldots}} \quad \text{or} \quad \boxed{x_{\text{cm}} = \frac{\sum_i m_i x_i}{\sum_i m_i}}$$

$$\boxed{y_{\text{cm}} = \frac{m_1 y_1 + m_2 y_2 + m_3 y_3 \ldots}{m_1 + m_2 + m_3 \ldots}} \quad \text{or} \quad \boxed{y_{\text{cm}} = \frac{\sum_i m_i y_i}{\sum_i m_i}}$$

$$\boxed{z_{\text{cm}} = \frac{m_1 z_1 + m_2 z_2 + m_3 z_3 \ldots}{m_1 + m_2 + m_3 \ldots}} \quad \text{or} \quad \boxed{z_{\text{cm}} = \frac{\sum_i m_i z_i}{\sum_i m_i}}$$

Properties of centre of mass

- Centre of mass is the point where the entire mass of a body is assumed to be concentrated
- Centre of mass may lie inside or outside the body
- Mass may or may not be present at the point of centre of mass
- Distance of a particle from COM is inversely proportional to mass of the particle
- COM lies closer the more massive region of the mass distribution
- Location of COM is independent of choice of origin
- Sum of moments of mass of the particles about their COM is zero
- State (i.e. position and motion) of centre of mass is unaffected by the internal forces acting on the system

Centre of mass of a continuous distribution of mass

Centre of mass of a system of continuous mass distribution is given by the relation(s)

$$x_{\text{cm}} = \frac{\int \mathrm{d}m\, x}{\int \mathrm{d}m}$$

$$y_{\text{cm}} = \frac{\int \mathrm{d}m\, y}{\int \mathrm{d}m}$$

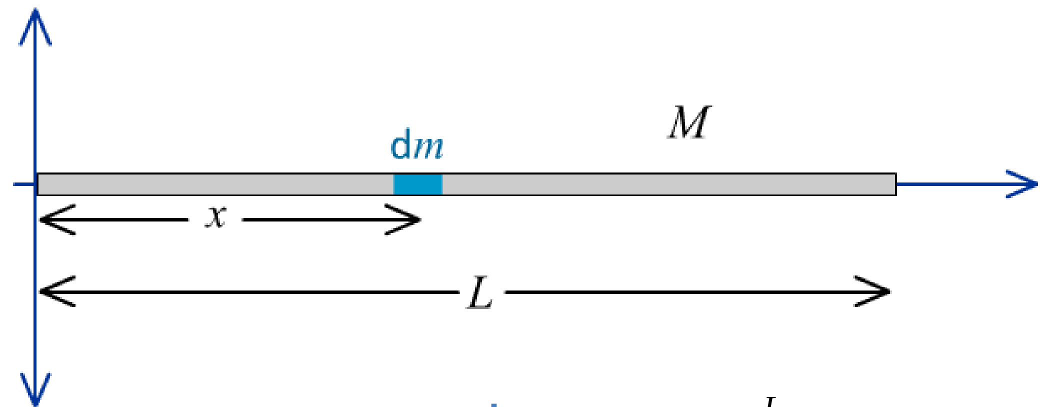
$$z_{\text{cm}} = \frac{\int \mathrm{d}m\, z}{\int \mathrm{d}m}$$

In he above relations, dm represents the mass of a small segment of the body located at a distance x or y or z from the origin.

The limits of integration depend on the distribution of mass about the origin.

Centre of mass of a thin wire of mass M and length L

Consider a thin straight wire of mass M and length L.
Consider a small segment of the wire of mass dm, at a distance x from the origin.



COM is given by the relation

$$x_{cm} = \frac{\int dm\ x}{\int dm} \quad \text{(i)}$$

where dm is given by

$$dm = \frac{M}{L}\,dx$$

Substitution this in eq (i)

$$x_{cm} = \frac{\int \frac{M}{L}\,dx\ x}{\int dm}$$

$$x_{cm} = \frac{\frac{M}{L}\int_0^L x^1\,dx}{M}$$

$$x_{cm} = \frac{M}{L}\frac{1}{M}\int_0^L x^1\,dx$$

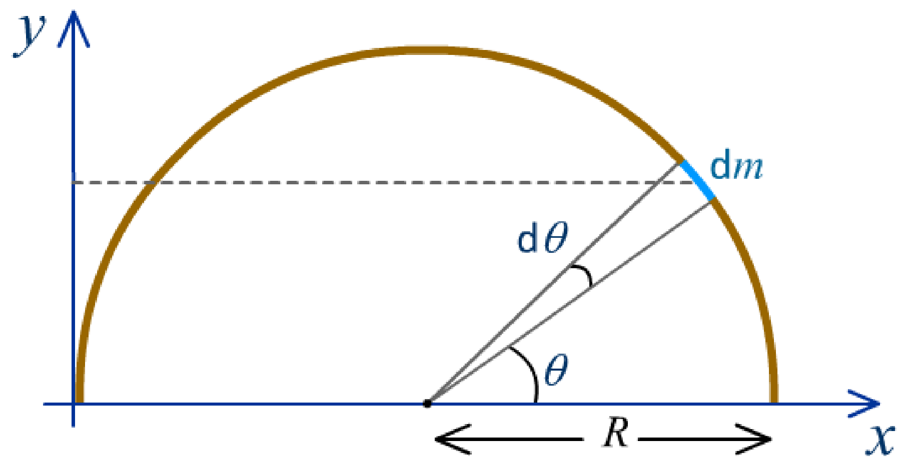
$$x_{cm} = \frac{1}{L}\int_0^L x^1\,dx$$

$$x_{cm} = \frac{1}{L}\left[\frac{x^2}{2}\right]_0^L$$

$$x_{cm} = \frac{1}{L}\left[\frac{L^2}{2} - \frac{0^2}{2}\right]$$

$$x_{cm} = \frac{L}{2}$$

Centre of mass of a thin semicircular wire of radius R



Consider a thin semicircular wire of mass M and radius R.

x-coordinate of COM is at R from the origin

Consider a small segment of the wire of mass $\mathrm{d}m$.

COM is given by the relation

$$\boxed{y_{cm} = \frac{\int \mathrm{d}m\ y}{\int \mathrm{d}m}} \quad \text{— i}$$

Denoting mass per unit length as λ, $\mathrm{d}m$ is given by

$$\mathrm{d}m = \lambda\ \mathrm{d}l$$

$$\mathrm{d}m = \lambda R\,\mathrm{d}\theta \quad \text{— ii}$$

$$y = R\sin(\theta) \quad \text{— iii}$$

Substituting (iii) and (ii) in eq (i)

$$y_{cm} = \frac{\int R\sin(\theta)\lambda\ R\,\mathrm{d}\theta}{M}$$

$$y_{cm} = \frac{\lambda R^2 \int \sin(\theta)\mathrm{d}\theta}{M}$$

$$y_{cm} = \frac{M}{\pi R}\frac{R^2 \int \sin(\theta)\mathrm{d}\theta}{M}$$

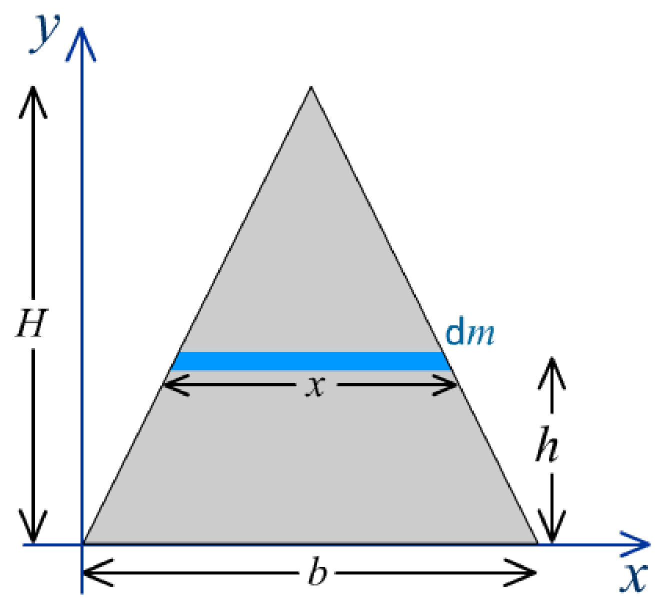
$$y_{cm} = \frac{R}{\pi}\left[-\cos(\theta)\right]_0^{\pi}$$

$$y_{cm} = \frac{R}{\pi}\left[1-(-1)\right]$$

$$y_{cm} = \frac{2R}{\pi}$$

Centre of mass of a triangular sheet base length b and height H

Consider a laminar triangular sheet of base b and height H. x-coordinate of COM is at $b/2$ from the origin. Consider a small segment of the sheet of mass dm.



COM is given by the relation

$$y_{cm} = \frac{\int dm\ y}{\int dm} \quad \text{i}$$

Denoting mass per unit area as σ, dm is given by

$$dm = \sigma\ dA$$

$$dm = \sigma\ x\,dh \quad \text{ii}$$

Considering similar triangles

$$\frac{H}{b} = \frac{H-h}{x}$$

$$x = \frac{(H-h)b}{H} \quad \text{iii}$$

Substituting this in eq(ii)

$$dm = \sigma \frac{(H-h)b}{H} dh$$

Substituting this in eq(i)

$$y_{cm} = \frac{\int \sigma \frac{(H-h)b}{H} dh\ h}{M}$$

$$y_{cm} = \frac{b\sigma}{M} \int \left(h - \frac{h^2}{H} \right) dh$$

Centre of mass of a triangular sheet base length *b* and height *H*

$$y_{cm} = \frac{b\sigma}{M}\int_0^H \left(h - \frac{h^2}{H}\right) dh$$

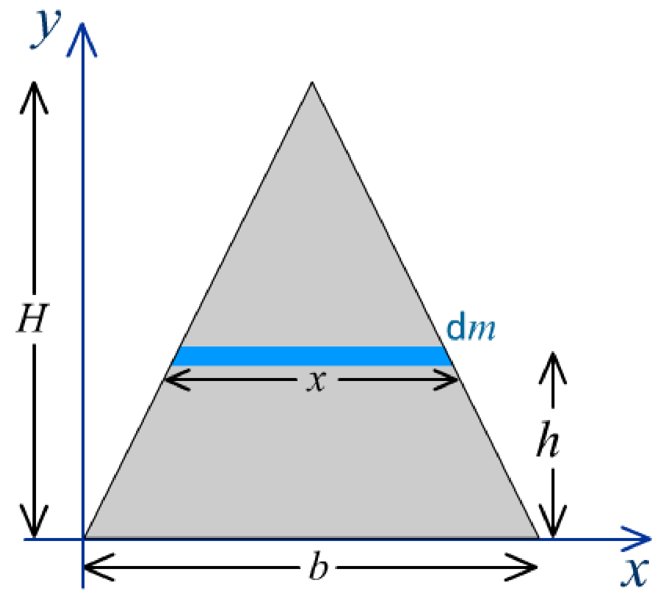
$$y_{cm} = \frac{b\sigma}{M}\left[\frac{h^2}{2} - \frac{h^3}{3H}\right]_0^H$$

$$y_{cm} = \frac{b\sigma}{M}\left[\frac{H^2}{2} - \frac{H^3}{3H}\right]_0^H$$

$$y_{cm} = \frac{b\sigma}{M}\frac{H^2}{6} \quad \text{iv}$$

Mass per unit area (σ) is given by

$$\sigma = \frac{M}{1/2bH} \quad \text{v}$$



Substituting this in eq (iv) we get

$$y_{cm} = \frac{b2M}{bH}\frac{H^2}{6}$$

$$y_{cm} = \frac{H}{3}$$

Centre of mass of combination of extended objects

- Centre of mass of a homogenous symmetric object lies at its geometric centre
- Each object my be represented by a point mass located at its centre of mass
- Centre of mass of the system consisting of such extended objects can then be obtained using the relation for point objects

Velocity of centre of mass

Centre of mass of a system consisting of discrete point masses is given by

$$\bar{r}_{\text{cm}} = \frac{m_1\bar{r}_1 + m_2\bar{r}_2 \ \ldots}{m_1 + m_2 \ \ldots}$$

Differentiating it w.r.t. time we get

$$\frac{d\bar{r}_{\text{cm}}}{dt} = \frac{d}{dt}\left[\frac{m_1\bar{r}_1 + m_2\bar{r}_2 \ \ldots}{m_1 + m_2 \ \ldots}\right]$$

$$\bar{v}_{\text{cm}} = \frac{1}{m_1 + m_2 \ldots}\frac{d}{dt}\left[m_1\bar{r}_1 + m_2\bar{r}_2 \ \ldots\right]$$

$$\bar{v}_{\text{cm}} = \frac{1}{m_1 + m_2 \ldots}\left[m_1\frac{d}{dt}\bar{r}_1 + m_2\frac{d}{dt}\bar{r}_2 \ldots\right]$$

$$\bar{v}_{\text{cm}} = \frac{1}{m_1 + m_2 \ldots}\left[m_1\bar{v}_1 + m_2\bar{v}_2 \ldots\right]$$

$$\boxed{\bar{v}_{\text{cm}} = \frac{\sum_i m_i\bar{v}_i}{\sum_i m_i}}$$

Acceleration of centre of mass

Velocity of centre of mass of a system is given by

$$\bar{v}_{cm} = \frac{m_1\bar{v}_1 + m_2\bar{v}_2 \ \ldots}{m_1 + m_2 \ \ldots}$$

Differentiating it w.r.t. time we get

$$\frac{d\bar{v}_{cm}}{dt} = \frac{d}{dt}\left[\frac{m_1\bar{v}_1 + m_2\bar{v}_2 \ \ldots}{m_1 + m_2 \ \ldots}\right]$$

$$\bar{a}_{cm} = \frac{1}{m_1 + m_2\ldots}\frac{d}{dt}\left[m_1\bar{v}_1 + m_2\bar{v}_2 \ \ldots\right]$$

$$\bar{a}_{cm} = \frac{1}{m_1 + m_2\ldots}\left[m_1\frac{d}{dt}\bar{v}_1 + m_2\frac{d}{dt}\bar{v}_2\ldots\right]$$

$$\bar{a}_{cm} = \frac{1}{m_1 + m_2\ldots}\left[m_1\bar{a}_1 + m_2\bar{a}_2\ldots\right]$$

$$\boxed{\bar{a}_{cm} = \frac{\sum_i m_i\bar{a}_i}{\sum_i m_i}}$$

Analysis of velocity and acceleration of COM

Velocity of centre of mass is given by

$$\bar{v}_{cm} = \frac{1}{m_1 + m_2 \ldots}\left[m_1\bar{v}_1 + m_2\bar{v}_2 \ldots\right]$$

$$(m_1 + m_2 \ldots)\ \bar{v}_{cm} = m_1\bar{v}_1 + m_2\bar{v}_2 \ldots$$

$$M\ \bar{v}_{cm} = m_1\bar{v}_1 + m_2\bar{v}_2 \ldots$$

$$M\ \bar{v}_{cm} = \bar{p}_1 + \bar{p}_2 \ldots$$

$$\boxed{\sum_i \bar{p}_i = M\ \bar{v}_{cm}}$$

Total linear momentum of a system of particles is equal to the product of mass of the system and velocity of its entre of mass.

Acceleration of centre of mass is given by

$$\bar{a}_{cm} = \frac{1}{m_1 + m_2 \ldots}\left[m_1\bar{a}_1 + m_2\bar{a}_2 \ldots\right]$$

$$(m_1 + m_2 \ldots)\bar{a}_{cm} = m_1\bar{a}_1 + m_2\bar{a}_2 \ldots$$

$$M\ \bar{a}_{cm} = m_1\bar{a}_1 + m_2\bar{a}_2 \ldots$$

$$M\ \bar{a}_{cm} = \bar{F}_1 + \bar{F}_2 \ldots$$

$$\boxed{\sum_i \bar{F}_i = M\ \bar{a}_{cm}}$$

Centre of mass of a system moves as if all the mass is concentrated at the COM and external force acts on centre of mass

Analysis of velocity and acceleration of COM

Velocity of centre of mass is given by

$$\bar{v}_{\text{cm}} = \frac{1}{m_1 + m_2 \ldots}\left[m_1\bar{v}_1 + m_2\bar{v}_2 \ldots\right]$$

$$(m_1 + m_2 \ldots)\ \bar{v}_{\text{cm}} = m_1\bar{v}_1 + m_2\bar{v}_2 \ldots$$

$$M\ \bar{v}_{\text{cm}} = m_1\bar{v}_1 + m_2\bar{v}_2 \ldots$$

$$M\ \bar{v}_{\text{cm}} = \bar{p}_1 + \bar{p}_2 \ldots$$

$$\boxed{\sum_i \bar{p}_i = M\ \bar{v}_{\text{cm}}}$$

Total linear momentum of a system of particles is equal to the product of mass of the system and velocity of its entre of mass.

Acceleration of centre of mass is given by

$$\bar{a}_{\text{cm}} = \frac{1}{m_1 + m_2 \ldots}\left[m_1\bar{a}_1 + m_2\bar{a}_2 \ldots\right]$$

$$(m_1 + m_2 \ldots)\bar{a}_{\text{cm}} = m_1\bar{a}_1 + m_2\bar{a}_2 \ldots$$

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$$M\ \bar{a}_{\text{cm}} = \bar{F}_1 + \bar{F}_2 \ldots$$

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Centre of mass of a system moves as if all the mass is concentrated at the COM and external force acts on centre of mass

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